Frobenius Theorem

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Chapter 1

Introduction

This project aims to really formalize the Frobenius Theorem from differential geometry in Lean 4.

Chapter 2

Frobenius Theorem

Consider a smooth manifold M of dimension m. For local questions we can take $M = \mathbb{R}^m$, which could correspond to a chart around some point $x_0 \in M$. All functions, vector fields and differential forms are presumed to be smooth (C^∞).

Definition 1 (involutivity)**.** Let $L_i = \sum_{k=1}^m f_i^k(x)\partial/\partial x^j$, $i = 1, \dots, r \le m$, be first order differential operators, such that the vector fields $v_i(x) = (f_i^k(x))_{k=1}^m$ are linearly independent. They are said to be in *involution* when there exist functions $c_{ij}^k(x)$ such that

$$L_i L_j - L_j L_i = \sum_{k=1}^r c_{ij}^k(x) L_k.$$

Theorem 2 (local Frobenius)**.** *If the first order differential operators L_i, $i = 1, \dots, r \le m$, are in involution, then there exist $m - r$ smooth functions $u^k(x)$ that satisfy the equations $L_i u^k(x) = 0$ and such that their gradients $\nabla u^k(x)$, $k = 1, \dots, m - r$ are linearly independent.*

Proof. This proof consists of chaining together several intermediate results, which are proven in separate lemmas below.

The first step is to replace the L_i operators by some better behaved operators $L'_{i'}$, namely satisfying $[L'_{i'}, L'_{j'}] = 0$ (Lem. 5(e)) and having a form adapted to a split local coordinate system (y, z) around x_0. The equations $L'_{i'} u = 0$ and $L_i u = 0$ are equivalent (Lem. 3, 4). Lem. 7 shows that there exists a new local coordinate system $(y, \bar{Z})$ on a neighborhood of x_0, where $\bar{Z}^k = \bar{Z}^k(y, z)$, which is better adapted to our differential equations. Lem. 6 actually shows that the contructed coordinates give us the desired solutions via $u^k(x) = \bar{Z}^k(y(x), z(x))$. □

Lemma 3. *If the L_i as in Def. 1 are in infolution, then the $L'_{i'} = \sum_{i=1}^r \alpha_{i'}^i L_i$, $i' = 1, \dots, r$, with smooth pointwise invertible $\alpha_{i'}^i$, are also in involution.*

Proof. It is sufficient to compute the commutator

$$\begin{aligned}
L'_{i'} L'_{j'} - L'_{j'} L'_{i'} &= \alpha_{i'}^i \alpha_{j'}^j (L_i L_j - L_j L_i) + \alpha_{i'}^i L_i(\alpha_{j'}^j) L_j - \alpha_{j'}^j L_j(\alpha_{i'}^i) L_i \\
&= \alpha_{i'}^i \alpha_{j'}^j \left(c_{ij}^k + (\alpha^{-1})_j^{j'} L_i(\alpha_{j'}^j) \delta_j^k - (\alpha^{-1})_i^{i'} L_j(\alpha_{i'}^i) \delta_i^k \right) L_k \\
&= \alpha_{i'}^i \alpha_{j'}^j \left(c_{ij}^k + (\alpha^{-1})_j^{j'} L_i(\alpha_{j'}^j) \delta_j^k - (\alpha^{-1})_i^{i'} L_j(\alpha_{i'}^i) \delta_i^k \right) (\alpha^{-1})_k^{k'} L'_{k'} \\
&= \sum_{k'=1}^r c'^{k'}_{i'j'} L'_{k'},
\end{aligned}$$

where the formula for $c'^{k'}_{i'j'}$ can be read off from the last equality. $\square$

Lemma 4. *For the operators $L'_{i'}$ as in Lem. 3, a smooth function u solves $L_i u = 0$, $i = 1, \dots, r$, iff it solves $L'_{i'} u = 0$, $i' = 1, \dots, r$.*

Proof. The computation

$$L'_{i'} u = \sum_{i=1}^{r} \alpha^i_{i'} (L_i u)$$

shows that $L_i u = 0$, $i = 1, \dots, r$, implies $L'_{i'} u = 0$ for any $i' = 1, \dots, r$. $\square$

Lemma 5. *For the operators L_i from Def. 1, given $x_0 \in M$, there exists an open coordinate neighborhood $U \ni x_0$ such that (a) there an exists invertible $\alpha^i_{i'}$ as in Lem. 3, (b) there exists a split local chart $(y, z)\colon M \supset U' \cong \mathbb{R}^r \times \mathbb{R}^{m-r}$, with (c) $(y(x_0), z(x_0)) = (0, 0)$, (d)*

$$L'_{i'} = \frac{\partial}{\partial y^{i'}} + \sum_{j=1}^{m-r} f^j_{i'}(y, z) \frac{\partial}{\partial z^j}.$$

and (e) $[L'_{i'}, L'_{j'}] = 0$, for $i', j' = 1, \dots, r$, which expressed in terms of $f^j_{i'}$ means (f)

$$\frac{\partial}{\partial y^{i'}} f^j_{j'} + \sum_{k=1}^{m-r} f^k_{i'} \frac{\partial}{\partial z^k} f^j_{j'} = \frac{\partial}{\partial y^{j'}} f^j_{i'} + \sum_{k=1}^{m-r} f^k_{j'} \frac{\partial}{\partial z^k} f^j_{i'}.$$

Proof. Start with the coordinates $(x^1, \dots, x^m)$ on U and consider the coordinate components $L_i = a^j_i(x) \frac{\partial}{\partial x^j}$. The rank of the matrix $a^j_i(x_0)$ must be r, otherwise the L_i vectors do not constitute a frame for the distribution $\mathcal{D}$. Hence, there exists a subset $I \subseteq \{1, \dots, r\}$ such that the matrix minor $(a^j_i)_{i \in I, 1 \le j \le m}$ is non-singular. Define the coordinates $y^{i'} = x^{I(i')} - (x_0)^{I(i')}$, $i' = 1, \dots, r$, and $z^j = x^{I^c(j)} - (x_0)^{I^c(j)}$, $j = 1, \dots, m - r$, where $I(i')$ and $I^c(j)$ is some ordering of the sets I and its complement I^c. Then, restrict to a sub-neighborhood $U'' \subseteq U$ that is split with respect to the (y, z) coordinates.

The new coordinate components are

$$L_i = \sum_{i'=1}^{r} a_i^{I(i')}(x(y, z)) \frac{\partial}{\partial y^{i'}} + \sum_{j=1}^{m-r} a_i^{I^c(j)}(x(y, z)) \frac{\partial}{\partial z^j}.$$

Let $\beta^{i'}_i(y, z) = a_i^{I(i')}(x(y, z))$ and $\gamma^j_i(y, z) = a_i^{I^c(j)}(x(y, z))$, so that by construction $\beta^{i'}_i(0, 0)$ is non-singular. Since $\beta\colon U'' \to \mathrm{Mat}(r, r)$ is smooth (hence a fortiriori continuous) and the subset of non-singular matrices in $\mathrm{Mat}(r, r)$ is open, there is a possibly smaller split sub-neighborhood $U' \subseteq U''$ on which β is everywhere non-singular. So, defining $\alpha^j_{i'}(y, z) = (\beta^{i'}_i(y, z))^{-1}$ on U'' satisfies the desired conclusions (a), (b), (c) and (d), where $f^j_{i'}(y, z) = \alpha^i_{i'}(y, z)\gamma^j_i(y, z)$.

To prove (e) and (f), consider the computation

$$\begin{aligned}
&[L'_{i'}, L'_{j'}] = L'_{i'} L'_{j'} - L'_{j'} L'_{i'} \\
&= \sum_{k'=1}^{r} c'^{k'}_{i'j'} L'_{k'} = \sum_{j=1}^{m-r} \left(\frac{\partial}{\partial y^{i'}} f^j_{j'} - \frac{\partial}{\partial y^{j'}} f^j_{i'} \right) \frac{\partial}{\partial z^j} + \sum_{k=1}^{m-r} \sum_{j=1}^{m-r} \left(f^j_{i'} \frac{\partial}{\partial z^j} f^k_{j'} - f^j_{j'} \frac{\partial}{\partial z^j} f^k_{i'} \right) \frac{\partial}{\partial z^k} \\
&= \sum_{j=1}^{m-r} \left(\frac{\partial}{\partial y^{i'}} f^j_{j'} + \sum_{k=1}^{m-r} f^k_{i'} \frac{\partial}{\partial z^k} f^j_{j'} - \frac{\partial}{\partial y^{j'}} f^j_{i'} - \sum_{k=1}^{m-r} f^k_{j'} \frac{\partial}{\partial z^k} f^j_{i'} . \right)
\end{aligned}$$

Hence, for each fixed i', j', the $\frac{\partial}{\partial y^{k'}}$ components of the right-hand side vanish, while those of the left-hand side equal $\sum_{k'=1}^{m-r} c'^{k'}_{i'j'} \frac{\partial}{\partial y^{k'}}$, meaning that all components of $c'^{k'}_{i'j'}$ must vanish, proving (e). On the other hand, the vanishing of the right-hand side of the last equality proves (f). $\square$

Lemma 6. *Consider the operators $L'_{i'}$ and the split neighborhood $U \ni x_0$ as in Lem. 5. Let $Z^k(y,z)$ and $\bar{Z}^k(y,z)$ satisfy the inversion identity $\bar{Z}^k(y, Z(y,z)) = z^k$, for all z on a sufficiently small neighborhood of $z = 0$, and for all y on a sufficiently small neighborhood of $y = 0$. Suppose that $Z^j(y,z)$ satisfies (a) $Z^j(0,z) = z^j$ and (b)*

$$\frac{\partial Z^j}{\partial y^{i'}}(y,z) = f^j_{i'}(y, Z^j(y,z)).$$

Then

$$L'_{i'} \bar{Z}(y,z) = 0, \quad i' = 1, \dots, r,$$

and vice versa.

Proof. Start by differentiating the inversion identity:

$$\begin{aligned}
0 = \frac{\partial}{\partial y^{i'}} z^k &= \frac{\partial}{\partial y^{i'}} \bar{Z}^k(y, Z(y,z)) \\
&= \frac{\partial \bar{Z}^k}{\partial y^{i'}}(y,z')\bigg|_{z'=Z(y,z)} + \frac{\partial Z^j}{\partial y^{i'}}(y,z)\, \frac{\partial \bar{Z}^k}{\partial z'^j}(y,z')\bigg|_{z'=Z(y,z)} \\
&= \left(\frac{\partial \bar{Z}^k}{\partial y^{i'}}(y,z') + f^j_{i'}(y,z') \frac{\partial \bar{Z}^k}{\partial z'^j}(y,z') \right)\bigg|_{z'=Z(y,z)} \\
&\quad + \left(\frac{\partial Z^j}{\partial y^{i'}}(y,z) - f^j_{i'}(y,z') \right) \frac{\partial \bar{Z}^k}{\partial z'^j}(y,z')\bigg|_{z'=Z(y,z)} .
\end{aligned}$$

Recall that being a diffeomorphism, the Jacobian $\frac{\partial \bar{Z}^k}{\partial z'^j}(y,z')$ non-singular on the sufficiently small split domain, with $(\frac{\partial \bar{Z}^k}{\partial z'^j}(y,z'))^{-1} = \frac{\partial Z^j}{\partial z^k}(y,z)\Big|_{z=\bar{Z}(y,z')}$. Hence, rearranging the last equality, we find

$$\frac{\partial Z^j}{\partial z^k}(y,z) L'_{i'} \bar{Z}^k(y,z')\bigg|_{z'=Z(y,z)} = -\left(\frac{\partial Z^j}{\partial y^{i'}}(y,z) - f^j_{i'}(y, Z(y,z)) \right).$$

Hence, if one side of the equality vanishes, then so does the other, which proves the desired equivalence. $\square$

Lemma 7. *Let $Z^j(y,z)$ be as in Lem. 6. Then, $\zeta^j(t,y,z) = Z^j(ty,z)$ satisfies $\zeta^j(0,y,z) = z^j$ and the equations*

$$\frac{\partial}{\partial t} \zeta^j(t,y,z) = y^{i'} f^j_{i'}(ty, \zeta(t,y,z)).$$

Conversely, if $\zeta^j(t,y,z)$ satisfies the initial value problem above, then there exists a sufficiently small neighborhood of $(y,z) = (0,0)$ for which $Z^j(ty,z)$ exists up to at least $t = 1$. Then $Z^j(y,z) = \zeta^j(1,y,z)$ satisfies the conditions in the hypotheses of Lem. 6.

Proof. The easy direction is proved by the following computation:

$$\begin{aligned}
\frac{\partial}{\partial t}\zeta^j(t,y,z) &= \frac{\partial}{\partial t}Z^j(ty,z) \\
&= y^{i'}\,\frac{\partial Z^j}{\partial y'^{i'}}(y',z)\bigg|_{y'=ty} \\
&= y^{i'}\,f^j_{i'}(y',Z^j(y',z))\big|_{y'=ty} \\
&= y^{i'} f^j_{i'}(ty,Z(ty,z)) = y^{i'} f^j_{i'}(ty,\zeta(t,y,z)).
\end{aligned}$$

For the converse direction, consider the following computation, where we use the ODE satisfied by $\zeta^j(t,y,z)$ and the identity from Lem. 5(f):

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial}{\partial y^{i'}}\zeta^j(t,y,z) - tf^j_{i'}(ty,\zeta(t,y,z))\right) &= \frac{\partial}{\partial y^{i'}}\frac{\partial}{\partial t}\zeta^j(t,y,z) \\
&\quad - f^j_{i'}(ty,\zeta(t,y,z)) - ty^{j'}\frac{\partial}{\partial y'^{j'}}f^j_{i'}(y',\zeta(t,y,z))\bigg|_{y'=ty} - t\left(\frac{\partial}{\partial t}\zeta^k(t,y,z)\right)\frac{\partial}{\partial z'^k} \\
&= \frac{\partial}{\partial y^{i'}}\left(y^{j'} f^j_{j'}(ty,\zeta(t,y,z))\right) \\
&\quad - f^j_{i'}(ty,\zeta(t,y,z)) - ty^{j'}\,\frac{\partial}{\partial y'^{j'}}f^j_{i'}(y',z')\bigg|_{y'=ty,z'=\zeta(t,y,z)} - ty^{j'}\,f^k_{j'}(y',z')\frac{\partial}{\partial z'^k} \\
&= \left(f^j_{i'}(y',z') + ty^{j'}\frac{\partial}{\partial y'^{i'}}f^j_{j'}(y',z') + \left(\frac{\partial}{\partial y^{i'}}\zeta^k(t,y,z)\right)y^{j'}\frac{\partial}{\partial z'^k}f^j_{j'}(y',z')\right)\bigg|_{y'=t} \\
&\quad - \left(f^j_{i'}(y',z') + ty^{j'}\frac{\partial}{\partial y'^{i'}}f^j_{j'}(y',z') + ty^{j'} f^k_{i'}(y',z')\frac{\partial}{\partial z'^k}f^j_{j'}(y',z')\right)\bigg|_{y'=ty,z'=\zeta} \\
&= \left(\frac{\partial}{\partial y^{i'}}\zeta^k(t,y,z) - tf^k_{i'}(ty,\zeta(t,y,z))\right)y^{j'}\frac{\partial}{\partial z'^k}f^j_{j'}(y',z')\bigg|_{y'=ty,z'=\zeta(t,y,z)}
\end{aligned}$$

Hence, we find that $\eta(t,y,z) = \frac{\partial}{\partial y^{i'}}\zeta^k(t,y,z) - tf^k_{i'}(ty,\zeta(t,y,z))$ satisfies a linear ODE. Hence, by the uniqueness of ODE solutions (Lem. 8(b)), if the initial condition $\eta(0,y,z)=0$ is satisfied, the solution must identically vanish, $\eta(t,y,z)=0$, which upon setting $t=1$ proves that $Z^j(y,z) = \zeta^j(1,y,z)$ satisfies the desired differential equation, (Lem. 8(c)). It remains to check the vanishing initial condition:

$$\begin{aligned}
\eta(0,y,z) &= \frac{\partial}{\partial y^{i'}}\zeta^j(0,y,z) - 0\cdot f^j_{i'}(0\zeta(0,y,z)) \\
&= \frac{\partial}{\partial y^{i'}}z^j - 0 = 0.
\end{aligned}$$

The proof is completed by noting that the inverse function $\bar{Z}(y,z)$ exists on a sufficiently small neighborhood of $(y,z)=(0,0)$, because the continuity of $\frac{\partial Z^j}{\partial z^k}$ and the property that $\frac{\partial Z^j}{\partial^k_z}\big|_{z=0} = \delta^j_k$ ensures that $Z^j(y,z)$ is an immersion (has non-singular jacobian) on a neighborhood of $(y,z) = (0,0)$ and hence a diffeomorphism on a possibly smaller neighborhood (use inverse function theorem). □

Lemma 8. *An ODE initial value problem (a sufficiently general one to cover the one for $\zeta^j(t,y,z)$ in Lem. 7 and the one for $\eta^j_{i'}(t,y,z)$ in the proof of Lem. 7) (a) has a solution that is jointly* *smooth in (t, y, z), which (b) is unique, and (c) exists (at least) up to time $t = 1$ on a sufficiently small neighborhood of $(y, z) = (0, 0)$.*

Proof. This should follow from the Picard-Lindelöf ODE existence and uniqueness theorem with parameters. □

Definition 9 (differential forms)**.**

Definition 10 (differential ideal)**.**

Theorem 11 (differential form Frobenius)**.** *If α_i, $i = 1, \dots k \le m - k$ are 1-forms on M that generate a closed differential ideal. Then there exist smooth scalar functions $u_i(x)$, $i = 1, \dots, m-k$ such that the exact 1-forms du_i, $i = 1, \dots, m - k$ generate the same differential ideal.*

Definition 12 (tangent distribution)**.** A *tangent distribution* on a manifold M is a vector subbundle $\mathcal{D} \hookrightarrow TM$ (equivalently, an embedding of vector bundles).

Definition 13 (Lie bracket)**.** On a manifold M, given two vector fields u, v (sections of the tangent bundle TM), their *Lie bracket* $w = [u, v]$ is the vector field that satisfies the identity $w(f) = u(v(f)) - v(u(f))$, where vector fields act as first order differential operators on a smooth function f. In coordinate form, if $u = u^i \partial_i$, $v = v^i \partial_i$, $w = w^i \partial_i$, then $w^j = u^i \partial_i v^j - v^i \partial_i u^i$. The vector fields u, v *commute* (or are *in involution* in the sense of Def. 1) if $[u, v] = 0$.

Definition 14 (involutive distribution)**.** A tangent distribution $\mathcal{D} \hookrightarrow TM$ is *involutive* if, for any two vector field sections u, v of $\mathcal{D}$, the Lie bracket $[u, v]$ is also a section of $\mathcal{D}$.

Definition 15 (integral submanifold)**.** Given a manifold M with a tangent distribution $\mathcal{D} \hookrightarrow TM$ of rank r (as a vector bundle), a submanifold $\iota \colon N \hookrightarrow M$ passing through $x_0 \in M$ is called an *integral submanifold* of the distribution $\mathcal{D}$ if it is everywhere tangent to $\mathcal{D}$, $T\iota(TN) \subseteq \mathcal{D}$, where naturally $\dim N \le r$. In the case $\dim N = r$, the integral submanifold is called *maximal (in dimension)*.

Definition 16 (foliation)**.**

Theorem 17 (vector field Frobenius)**.** *Let $\mathcal{D} \subseteq TM$ be an involutive tangent space distribution of rank $r \le m = \dim M$. Then, for every $x_0 \in M$, there exists a maximal integral submanifold $\iota \colon \mathbb{R}^n \hookrightarrow M$ of $\mathcal{D}$ such that $\iota(0) = x_0$. Moreover, these integral submanifolds collect into a r-dimensional foliation of M whose leaves are everywhere tangent to the distribution $\mathcal{D}$.*